Math 323 S Probability Theory

3/18/2023

Convergence in square mean (L^2)

$X_n \to X$ in square mean

$$\lim_{n\to\infty} \mathbb{E}\left((X_n - X)^2 \right) = 0$$

Convergence in probability

$X_n \xrightarrow{P} X$ if

$$\forall \delta > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \delta) = 0$$

Convergence in distribution

$X_n \Rightarrow X$

$$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x)$$

whenever $F(x) = \mathbb{P}(X \leq x)$ is continuous at x.

$$S_n = \frac{1}{N} \sum_{i=1}^{N} X_i \qquad X \quad \text{i.i.d.}$$

$$\mathbb{E}(X_i) = 0 \qquad \text{var}(X_i) = 1$$

$$\mathbb{P}(|S_n| > \delta) \leq \frac{\text{Var}(S_n)}{\delta^2}$$

$$\leq \frac{1}{N \delta^2}$$

Other example

$$S_n \simeq \mathcal{N}\left(0, \frac{1}{N}\right)$$

$$\mathbb{P}(|S_n| > 0) = 1$$

$$\mathbb{P}(|S_n| > \delta) = \frac{1}{N\delta^2}$$

$$S_n = \frac{1}{N} \sum_i X_i \qquad X_i \quad \text{i.i.d}$$

$$S_n \to \mathbb{E}(X_i) \quad \text{in square mean}$$

Convergence in square mean $\Longrightarrow$ convergence in probability.

Weak Law of Large Numbers.

$S_n \to \mathbb{E}(X_i)$ in prob.

Theorem if $X_n \to X$ in probability the $X_n \Longrightarrow X$ in distribution.

Proof:

$$F_n(x) \to F(x) \quad n \to \infty$$

$$F_n(x) = \mathbb{P}(X_n \leq x) =$$

$$\mathbb{P}(X_n \leq x \ \& \ X \leq x+\varepsilon) +$$

$$\mathbb{P}(X_n \leq x \ \& \ X \geq x+\varepsilon) \leq$$

$$\mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$$

$$F_n(x) \leq F(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$$

$$F(x-\varepsilon) \leq F_n(x) + \mathbb{P}(|X_n - X| > \varepsilon)$$

$\forall \varepsilon$

$$F_n(x) \leq F(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$$

$$F_n(x) \geq F(x-\varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon)$$

limit for N to infinity

$\forall \varepsilon$

$$F(x-\varepsilon) \leq \lim_{n\to\infty} F_n(x) \leq F(x+\varepsilon)$$

$\varepsilon \to 0$

$$\lim_{n\to\infty} F_n(x) = F(x)$$

Square mean

$\Downarrow$

Probability

$\Downarrow$

Distribution

Distribution $\not\Rightarrow$ Probability

$$X_0 = \begin{cases} 1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2} \end{cases}$$

$$X_n = (-1)^n X_0$$

$F_n(x)$ is the same for every n.

$P(|X_n - X_0| > 1) = 1$ if n is odd

No convergence in Prob.

If $X_n \Rightarrow c$ Then $X_n \xrightarrow{P} c$

$F_n(x) \to 0 \qquad x < c$

$F_n(x) \to 1 \qquad x > c$

$$\mathbb{P}(|X_n - c| > \delta) = \mathbb{P}(X_n - c > \delta) + \mathbb{P}(X_n - c < -\delta) =$$

$$\underbrace{F_n(c-\delta)}_{\to 0} + \underbrace{(1 - F_n(c+\delta))}_{\to 0}$$

$$\lim_{n \to \infty} \mathbb{P}(|X_n - c| > \delta) = 0$$

X_i are i.i.d. with $\mathbb{E}(X_i) = \mu$

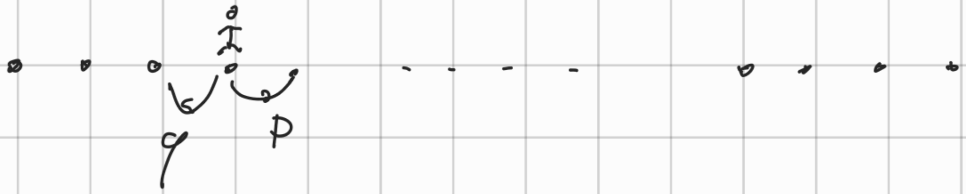
$$S_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

Then

$$S_n \Rightarrow \mu$$

and so

$$S_n \xrightarrow{P} \mu$$



$q = 1 - p$